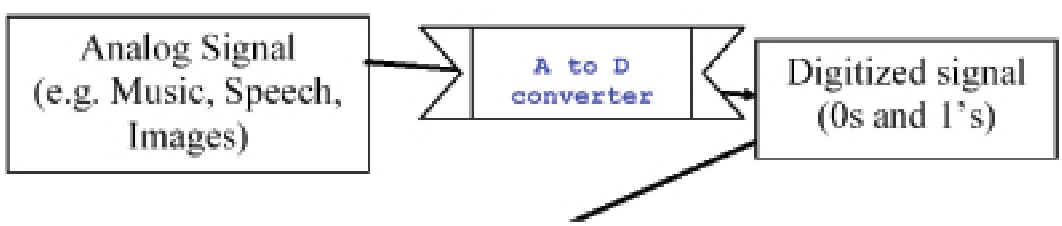
SOPHOMORE COLLEGE
MATHEMATICS OF THE INFORMATION AGE

SECRETS OF THE UNIVERSE, PART 1
FOURIER SERIES

In our rapid treatment of communications we arrived at what I (earnestly) referred to as a Major Secret of the Universe:

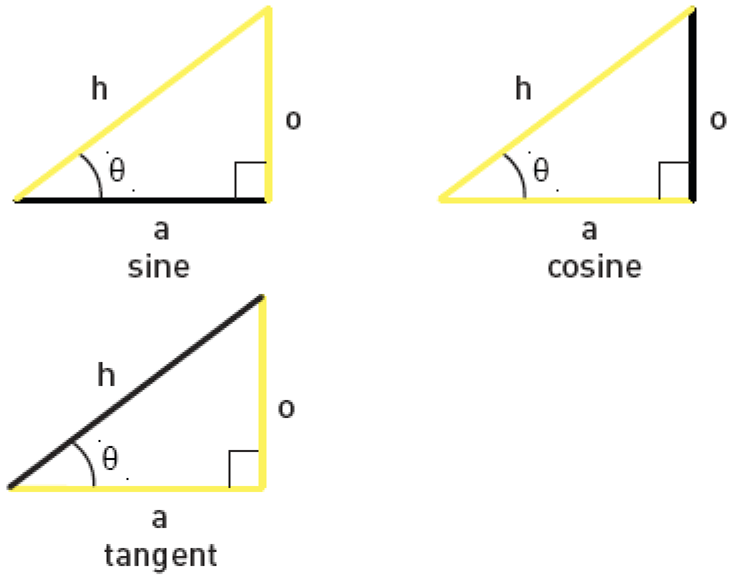
Every signal has a spectrum. The spectrum determines the signal.

Learning this secret is learning about Fourier analysis. The subject is both vast and deep, and its origins are far removed from where we're headed: our goal at present is to use Fourier analysis to accomplish the very first task of the course syllabus.

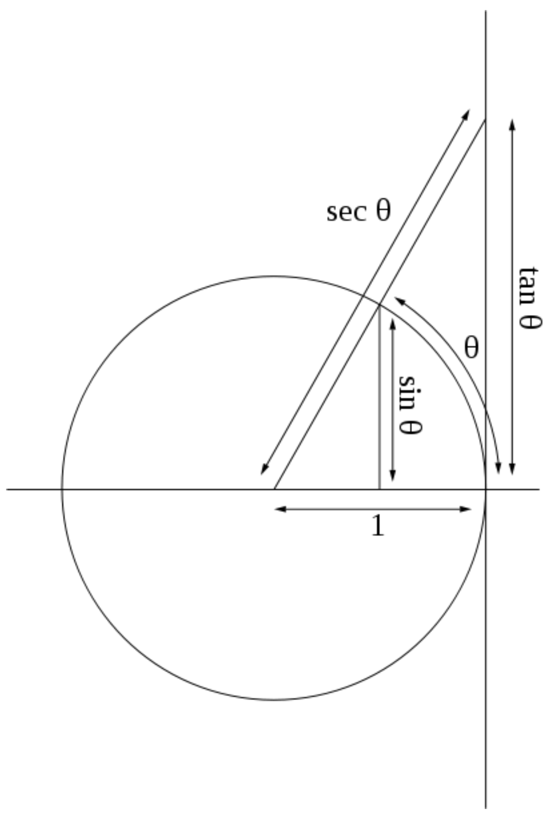


An analog signal is typically an electronic version of a signal that we encounter with our senses, a sound or an image, for example. We want to 'digitize' that signal, to turn it into a list of 0's ands 1's, in a way that captures the information in the signal sufficiently well so that, after much in between, when we ultimately turn it back to analog our perceptions of the signal are acceptable. Fourier analysis is the key to this.

I want to start the description of ingredients of signals at a very unlikely place, triangles. When you studied triangles and the measurement of triangles you encountered for the first time the sine, cosine and tangent.

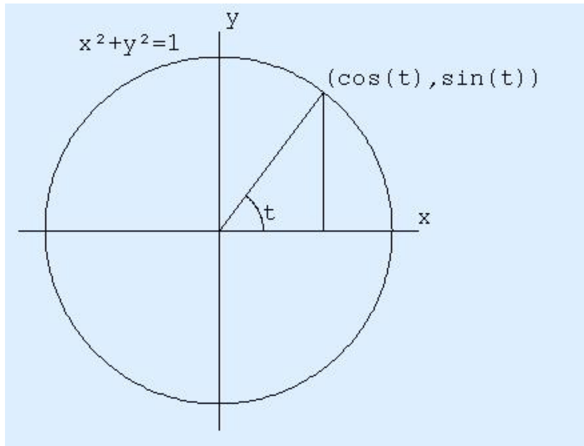


These basic quantities are defined through ratios of lengths. There is another way to express the tangent, making its name particularly fitting.



We see here, as we did not see in the original picture of a triangle, a connection between the trigonometric ratios and a circle. It is worth making this connection more explicit. The trigonometric quantities $\cos\theta$ and $\sin\theta$ were defined only for angles in a right triangle, that is, only for angles between 0 and a right angle $\pi/2$ (we'll use radian measure for angles).

But that picture of a triangle is part of a picture within part of a circle, and that part of a circle is within an entire circle. We see the possibility of extending the definition of sine and cosine for angles beyond $\pi/2$.



This is a *new definition* of sine and cosine. It's not that the original definition in terms of a right triangle is wrong, is that it's possible to give a new definition that includes the old and goes beyond it.

What has been gained? First, there is the gain in defining $\cos\theta$ and $\sin\theta$ for all angles between 0 and 2π. But more than that, much more, is the realization that sine and cosine, of their nature, *repeat their values.*

- Picture of circle with θ and $\theta + 2\pi$ identified, hence the relationship

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sin(\theta + 2\pi) = \sin\theta$$

One says that the sine and cosine are *periodic of period* 2π. It follows from this that

$$\cos(\theta + 2\pi k) = \cos\theta$$
$$\sin(\theta + 2\pi k) = \sin\theta$$

for $k = 0, 1, 2, \ldots$. (Proof of this?)

From the interpretation of $\cos\theta$ and $\sin\theta$ as giving a point on the unit circle, we move toward a dynamic picture and think of a moving point.

- Rotate counterclockwise around the circle to get graphs of $\sin\theta$ and $\cos\theta$ for $\theta \geq 0$.
- What about $\theta \leq 0$?

We extend the definition of sine and cosine still further to include *negative* angles. We then have $\sin\theta$ and $\cos\theta$ defined for all real values of θ and we can say

$$\cos(\theta + 2\pi k) = \cos\theta$$
$$\sin(\theta + 2\pi k) = \sin\theta$$

for $k = 0, \pm 1, \pm 2, \ldots$.

The periodicity of sine and cosine is absolutely fundamental – and absolutely not news to you. It shouldn't be necessary to try to sell periodicity as an important physical (and mathematical) phenomenon – you've seen examples and applications of periodic behavior in probably (almost) every science class you've taken. I would only remind you that periodicity often shows up in two varieties, sometimes related, sometimes not. Generally speaking we think about periodic phenomena according to whether they are *periodic in time* or *periodic in space*.

In the case of time the phenomenon comes to you, so to speak. For example, you stand at a fixed point in spaces and a note reaches your ear as a (longitudinal) pressure wave, a periodic compression and rarefaction of the air. In the case of space, you come to the phenomenon. You take a picture and you observe repeating patterns. One important way this occurs is when there is a repeating pattern or some kind of symmetry in a spatial region, and physically observable quantities associated with that region have a repeating pattern that reflects this. For example, a crystal has a regular, repeating pattern of atoms. The electron density distribution is then a periodic function of the spatial variable (in $\mathbf{R}^3$) that describes the crystal. Temporal and spatial periodicity come together most naturally in wave motion, where periodicity in time is frequency and periodicity in space is wavelength (if we take a snapshot of the wave to fix the time). So now we have mathematical functions that exhibit the ubiquitous, important natural phenomenon of periodicity.

Finally, though in the examples above I said "a point going around a circle ...", I didn't say how fast the point is supposed to be going around, so to complete the move toward a more dynamic view of sine and cosine let's take this up. It's convenient, when thinking dynamically, to use the phrase 'one cycle' for one complete trip around the circle, starting wherever (as specified by the phase) and ending at the point where we started. It's then also convenient to take as a standard the motion where the point completes one cycle in one second. How do we modify the sine and cosine to achieve this? Let t represent time. Then as t goes from 0 to 1 we want the angle θ to go from 0 to 2π The easiest thing is to make θ related to t by

$$\theta = 2\pi t\,,$$

and so the modified sine and cosine (with phase 0) are

$$\cos 2\pi t \quad \text{and} \quad \sin 2\pi t\,, \quad 0 \le t \le 1\,.$$

These functions together with the restriction that $0 \le t \le 1$ describe a point moving counterclockwise once around the unit circle in one second, starting and ending at the point $(1, 0)$.

- What are the graphs of $\cos 2\pi t$ and $\sin 2\pi t$?

If we want to start at a different point, specified by a phase ϕ, we'd consider

$$\cos(2\pi t + \phi) \quad \text{and} \quad \sin(2\pi t + \phi)\,.$$

If we wanted to modify the amplitude to A we'd form

$$A\cos(2\pi t + \phi) \quad \text{and} \quad A\sin(2\pi t + \phi)\,.$$

What functions describe a point completing ν cycles in 1 second? Here as t goes from 0 to 1 we want θ to go from 0 to $2\pi\nu$. To achieve this we simply take

$$\theta = 2\pi\nu t$$

So, for example, if $\nu = 2$ we make two complete cycles in 1 second, while if $\nu = 1/2$ we only make a half-cycle (half way around the circle) in one second. The modified sine and cosine are, accordingly,

$$\cos 2\pi\nu t \quad \text{and} \quad \sin 2\pi\nu t\,.$$

- What are the graphs of $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$, for $\nu = 2$, $\nu = 1/2$?
- What is the period of $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$, for $\nu = 2$, $\nu = 1/2$?

The number ν in $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$ is called the *frequency*. It has units cycles per second and dimension 1/sec. This unit is called the *Hertz*, abbreviated *Hz*, after Heinrich Hertz (1857 - 1894), who did important experimental work on the existence of electromagnetic waves, verifying the theoretical work of James Clerk Maxwell.

The relationship between the period and frequency for $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$ is

$$\text{period} = \frac{1}{\text{frequency}}\,.$$

We saw this in the graphs, and we can also see it algebraically, for

$$\cos(2\pi\nu(t + \frac{1}{\nu})) = \cos(2\pi\nu t + 2\pi\nu\frac{1}{\nu}) = \cos(2\pi\nu t + 2\pi) = \cos(2\pi\nu t)$$

with the same kind of calculation showing that

$$\sin(2\pi\nu(t + \frac{1}{\nu})) = \sin 2\pi\nu t\,.$$

High frequency means short (small) period – many cycles in one second. Low frequency means long (large) period – few cycles in one second.

Allowing for a change of frequency, amplitude, and phase, the basic signal processing we do to a sine and cosine is to form

$$A\cos(2\pi\nu t + \phi) \quad \text{and} \quad A\sin(2\pi\nu t + \phi)\,.$$

So much for modifications of a single sine or a single cosine. Interesting enough. What if we combine sines and cosines of *different* frequencies, and amplitudes, and phases? That's when things get *really* interesting.

It all adds up

I want to spend a little time on the algebra of summing trig functions. First of all, it's convenient to assume that the fundamental frequency is 1 and to add in integral frequencies together with possible phase shifts. One way of writing a general finite sum of such sinusoids is then

$$\sum_{n=1}^{N} A_n \sin(2\pi nt + \phi_n).$$

It will take us a little time to understand why this is the 'general' sort of sum of trig functions that's useful to consider, *i.e.* why we look at integer multiples of the fundamental period.

What about the phase shifts for each term? That certainly adds a degree of generality, mathematically, but it's also important, practically, to include them. At one end of a channel we generate a signal, possibly one with many frequencies in its spectrum. Then, for instance, in the case of a family of radiating antennas the signals from each may be out of phase with each other, and we must take that into account. Even if the signal starts out with all sinusoidal parts in phase it may not end up that way. As the signal travels through a channel different parts of the signal may follow different paths.[1]. When the signal reaches

[1] In some cases this can depend on the frequency. This happens for light propagating through an optical fiber. To minimize dispersion one wants monochromatic light – one frequency

the end of the channel the different sinusoidal terms may then be out of phase – they still have the same frequencies as when they started, but they may not all be arriving at the same time. If we want to write down an expression that models the signal at the end of its journey we should account for the phase changes.

Having said so much about the aspects of this form of the sum, there are other ways of writing a general sum of trig functions that prove to be easier to handle algebraically. Use the addition formula for the sine function

$$\sin(2\pi t + \phi_n) = \sin(2\pi nt)\cos\phi_n + \cos(2\pi nt)\sin\phi_n$$

to write an alternative form for the sum as

$$\begin{aligned} \sum_{n=1}^{N} A_n \sin(2\pi nt + \phi_n) &= \sum_{n=1}^{N} A_n(\sin(2\pi nt)\cos\phi_n + \cos(2\pi nt)\sin\phi_n) \\ &= \sum_{n=1}^{N}(a_n \cos 2\pi nt + b_n \sin 2\pi nt). \end{aligned}$$

Here

$$a_n = A_n \sin\phi_n, \quad b_n = A_n \cos\phi_n.$$

Thus a general sum of sinusoids can also be written as a sum of sines and cosines, with no phase shifts;

$$\sum_{n=1}^{N}(a_n \cos 2\pi nt + b_n \sin 2\pi nt),$$

Note that

$$\sqrt{a_n^2 + b_n^2} = |A_n|,$$

so the amplitudes A_n from the first form of the sum are still 'visible' in the second form of the sum. What's not as visible is what happened to the phases in going from the first form to the second, though they're still there in the a_n and b_n. Once again, the fact that there are both sines and cosines in the second form of the sum is due to the phase shifts in the first.

I've neglected to say that it's often good to include a constant term. Graphically, this just shifts the whole picture up (or down) We thus write the most general sum as

$$\frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos 2\pi nt + b_n \sin 2\pi nt)$$

The reason for writing the constant term as a fraction is a convention that makes some subsequent notations easier; we'll see.

In many ways the sine - cosine form of the sum is more convenient to work with than the sum of phase-shifted sine functions. But the most convenient form of all uses complex numbers and complex exponentials, to represent the terms.

I'm going to assume that you're comfortable with the basic algebra and geometry of complex numbers, and there are separate notes that review some aspects. For our present work, the important formula (the only important formula) is Euler's formula:

$$e^{it} = \cos t + i \sin t.$$

Euler's formula has as a consequence the most wondrous equation in all of mathematics:

$$e^{i\pi} + 1 = 0\,.$$

One other important thing to observe is that if m is any integer then

$$e^{2\pi i m} = \cos 2\pi m + i \sin 2\pi m = 1.$$

and

$$e^{\pi i m} = \begin{cases} 1, & m \text{ even} \\ -1, & m \text{ odd} \end{cases}$$

We'll use this later.

Substituting $2\pi i n t$ for t in

$$e^{it} = \cos t + i \sin t,$$

we can write

$$e^{2\pi n i t} = \cos 2\pi n t + i \sin 2\pi n t.$$

We can solve separately for the cosine and the sine (the real and the imaginary parts) by first using:

$$e^{-2\pi i n t} = \cos(-2\pi n t) + i \sin(-2\pi n t) = \cos 2\pi n t - i \sin 2\pi n t.$$

and then adding and subtracting the equations. This gives

$$\cos 2\pi n t = \frac{e^{2\pi n i t} + e^{-2\pi i n t}}{2}, \quad \sin 2\pi n t = \frac{e^{2\pi n i t} - e^{-2\pi i n t}}{2i} = -i\frac{e^{2\pi n i t} - e^{-2\pi i n t}}{2},$$

where we have also used the fact that $1/i = -i$. For the real excitement, plug this into the sum of sines and cosines:

$$\begin{aligned} &\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos 2\pi n t + b_n \sin 2\pi n t) = \\ &\frac{a_0}{2} + \sum_{n=1}^{N} \left(-i b_n \frac{e^{2\pi n i t} - e^{-2\pi i n t}}{2} + a_n \frac{e^{2\pi n i t} + e^{-2\pi i n t}}{2} \right) = \\ &\frac{a_0}{2} + \sum_{n=1}^{N} \left(\frac{1}{2}(a_n - i b_n) e^{2\pi n i t} + \frac{1}{2}(a_n + i b_n) e^{-2\pi i n t} \right). \end{aligned}$$

We can put this into still a more compact form. We can think of the big sum, above, as two sums:

$$\sum_{n=1}^{N} \left(\frac{1}{2}(a_n - i b_n) e^{2\pi n i t} + \frac{1}{2}(a_n + i b_n) e^{-2\pi i n t} \right) = \sum_{n=1}^{N} \frac{1}{2}(a_n - i b_n) e^{2\pi n i t} + \sum_{n=1}^{N} \frac{1}{2}(a_n + i b_n) e^{-2\pi i n t},$$

Now, we want to write these as a sum over 'positive n' and 'negative n'. First, put

$$c_n = \frac{1}{2}(a_n - i b_n), \quad \text{and} \quad c_{-n} = \frac{1}{2}(a_n + i b_n).$$

(Thus c_{-n} is the complex conjugate of c_n, written $c_{-n} = \overline{c_n}$.) Write $c_0 = a_0/2$; *here comes the reason for writing the constant term as* a_0 *over* 2. Leave the first sum alone, and write the second sum with n going from -1 to $-N$. Then the expression becomes

$$\frac{a_0}{2} + \sum_{n=1}^{N} c_n e^{2\pi nit} + \sum_{n=-1}^{-N} c_n e^{2\pi nit}.$$

We can combine everything to write

$$\frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos 2\pi nt + b_n \sin 2\pi nt) = \sum_{n=-N}^{N} c_n e^{2\pi nit}.$$

Truly, the most compact and useful way of writing a sum of trig functions, of the type we've considered, is in the complex form:

$$\sum_{n=-N}^{N} c_n e^{2\pi nit}, \quad c_{-n} = \overline{c_n}.$$

You may be upset at writing something real (a signal) in terms of something complex. My only advice to you is to *get over it*. The relation $c_{-n} = \overline{c_n}$ between coefficients is important and don't forget about it. A *real* signal must have this property when expressed in terms of complex exponentials.

For now we think in terms of time and frequency. Suppose I have a complicated looking real, periodic signal. I can scale the time to assume that the pattern repeats every 1 second (for example from $t = 0$ to $t = 1$). Call the signal $f(t)$.

Can we write it in the form

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi nit}\,?$$

The unknowns in this expression are the coefficients c_n. Can we find them?

Note that writing a real signal this way already imposes the condition $c_{-n} = \overline{c_n}$ on the coefficients.

Let's take the coefficient c_k, for some k. We can isolate it by multiplying through both sides by $e^{-2\pi ikt}$:

$$\begin{aligned} e^{-2\pi ikt} f(t) &= e^{-2\pi ikt} \sum_{n=-N}^{N} c_n e^{2\pi nit} \\ &= \cdots + e^{-2\pi ikt} c_k e^{2\pi ikt} + \cdots \\ &= \cdots + c_k + \cdots \end{aligned}$$

$$c_k = e^{-2\pi ikt} f(t) - \sum_{n=-N, n\neq k}^{N} c_n e^{-2\pi ikt} e^{2\pi nit} = e^{-2\pi ikt} f(t) - \sum_{n=-N, n\neq k}^{N} c_n e^{2\pi i(n-k)t}.$$

That pulls out the coefficient c_k, but the expression involves all of the other unknown coefficient. Another idea is needed. That idea is *integrating* both sides from 0 to 1.

Just as in calculus, we can evaluate the integral by

$$\begin{aligned}
\int_0^1 e^{2\pi i(n-k)t}\,dt &= \frac{1}{2\pi i(n-k)} e^{2\pi i(n-k)t}\Big]_{t=0}^{t=1} \\
&= \frac{1}{2\pi i(n-k)}(e^{2\pi i(n-k)} - e^0) \\
&= \frac{1}{2\pi i(n-k)}(1-1) \\
&= 0.
\end{aligned}$$

Note that we needed to know $n \neq k$ here.

Since the integral of the sum is the sum of the integrals, and the coefficients c_n come out of each integral, we have a formula for the k'th coefficient (for each k):

$$c_k = \int_0^1 e^{-2\pi i k t} f(t)\,dt.$$

Observe how the symmetry relation $c_{-n} = \overline{c_n}$ comes out of this.

$$\overline{c_k} = \overline{\int_0^1 e^{-2\pi i k t} f(t)\,dt} = \int_0^1 e^{2\pi i k t} f(t)\,dt = c_{-k}.$$

This calculation works because $f(t)$ is real. Observe also that the zeroth coefficient is the average value of the function:

$$c_0 = \int_0^1 f(t)\,dt.$$

Let's summarize and be careful to note what we've done here – and what we haven't done. We've shown that *if* we can write a periodic function $f(t)$ as a sum

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t}, \quad c_{-n} = \overline{c_n},$$

then the coefficients have to be given by

$$c_n = \int_0^1 f(t) e^{-2\pi i n t}\,dt.$$

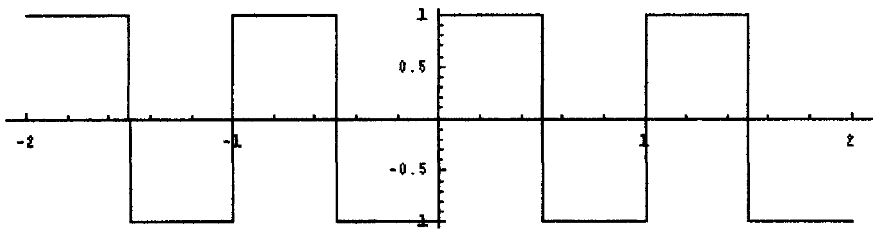
These are called the *Fourier coefficients* of $f(t)$, named after Joseph Fourier who introduced these ideas into mathematics and science. The sum

$$\sum_{n=-N}^{N} c_n e^{2\pi i n t}$$

is called the (finite) *Fourier series* for $f(t)$. If you want to be mathematically hip and impress your friends at cocktail parties, use the notation

$$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t}\,dt$$

for the Fourier coefficients.



Let's show one more property of the Fourier coefficients, that they can be calculated by intergrating over *any* interval of length 1, i.e., over any cycle. For this consider the function of a defined by

$$\phi(a) = \int_a^{a+1} e^{-2\pi int} f(t)\, dt.$$

Differentiate with respect to a:

$$\begin{aligned}
\phi'(a) = e^{-2\pi in(a+1)} f(a+1) - e^{-2\pi ina} f(a) \\
&= e^{-2\pi ina} e^{-2\pi in} f(a+1) - f(a) \\
&= e^{-2\pi ina} (f(a+1) - f(a)) \\
&= 0 \quad \text{because } f(t) \text{ is periodic of period 1.}
\end{aligned}$$

Thus $\phi'(a)$ is identically zero and hence $\phi(a)$ is constant – the value does not depend on a. In particular,

$$\int_a^{a+1} e^{-2\pi int} f(t)\, dt = \phi(a) = \phi(0) = \int_0^1 e^{-2\pi int} f(t)\, dy = \hat{f}(n).$$

OK, all this is fine. But I *have not* shown that you actually have the equality

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi nit}, \quad c_{-n} = \overline{c_n},$$

that you *can* actually write $f(t)$ as such a sum. Think it's true? How many terms might you have to take? What if you take an infinite series?

Let's look at an example. Consider a 'square wave' of period 1, such as illustrated, below. Let's calculate the Fourier coefficients. The function is

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases}$$

and then extended to be periodic. The zeroth coefficient is the average value of the function on $0 \le t \le 1$. Obviously this is zero. For the other coefficients we have

$$\begin{aligned}\hat{f}(n) &= \int_0^1 e^{-2\pi int} f(t)\,dt \\ &= \int_0^{1/2} e^{-2\pi int}\,dt - \int_{1/2}^1 e^{-2\pi int}\,dt \\ &= \left[-\frac{1}{2\pi in} e^{-2\pi int}\right]_0^{1/2} - \left[-\frac{1}{2\pi in} e^{-2\pi int}\right]_{1/2}^1 \\ &= \frac{1}{\pi in}(1 - e^{-\pi in})\end{aligned}$$

According to this we should consider the *infinite* Fourier series

$$\sum_{n \neq 0} \frac{1}{\pi in}(1 - e^{-2\pi in}) e^{2\pi int}$$

After the fact we might reflect that, of course we shouldn't expect to be able to use a finite Fourier series to represent a square wave. The complex exponentials are continuous and so is a finite sum of them, while the square wave has jump discontinuities.

We can write the sum in a simpler form by first noting that

$$1 - e^{-\pi in} = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

so the series becomes

$$\sum_{n \text{ odd}} \frac{2}{\pi in} e^{2\pi int}.$$

Now combine the positive and negative terms and use

$$e^{2\pi int} - e^{-2\pi int} = 2i \sin 2\pi nt.$$

Substituting this into the series and writing $n = 2k + 1$, our final answer is

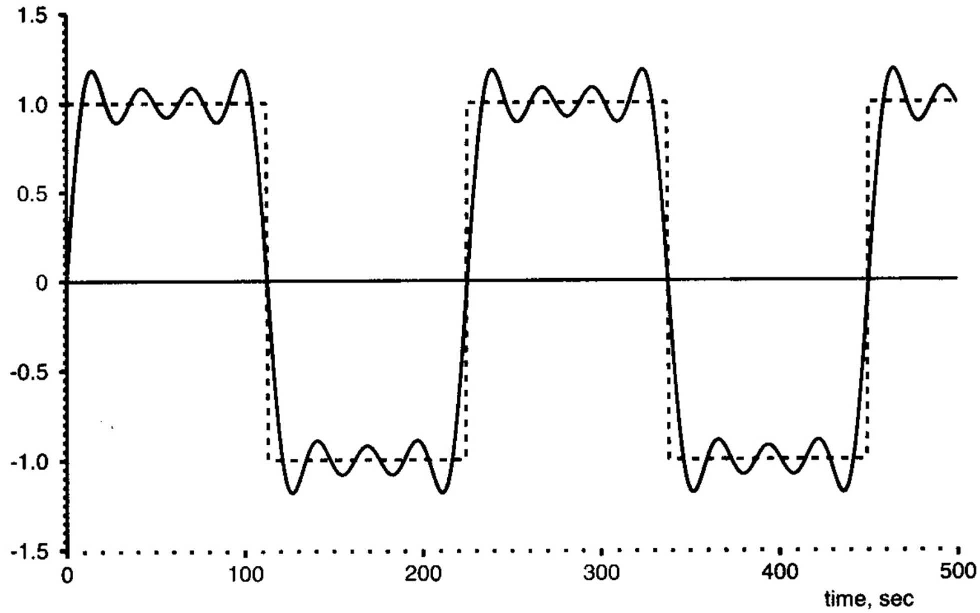
$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t.$$

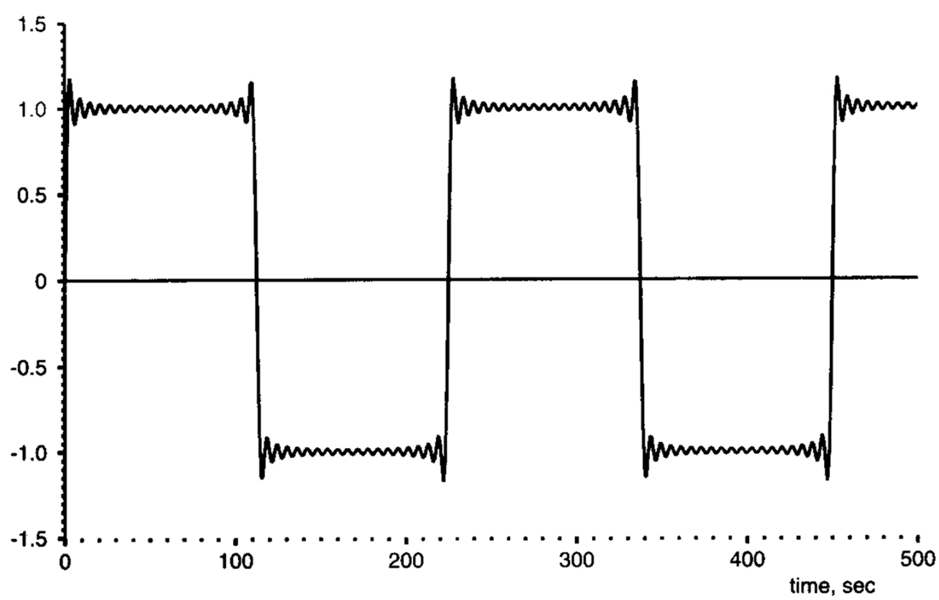
(Note that the function $f(t)$ is *odd* and this jibes with the Fourier series having only sine terms.)

What kind of series is this? In what sense does it converge, if at all, and can we write

$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t\,?$$

If we look at the graphs of partial sums we see some strange phenomena. The graphs below are sums of 9 and 39 terms, respectively.


1.5
1.0
0.5
0
-0.5
-1.0
-1.5
0
100
200
300
400
500
time, sec


1.5
1.0
0.5
0
-0.5
-1.0
-1.5
0
100
200
300
400
500
time, sec

To relive the struggles that went into understanding the convergence of Fourier series and the proper sense of equality between the function and its Fourier series, such as that for the square wave, is to relive the history of much of mathematical analysis in the 19th and early 20th centuries. We won't, but I do think it is worth the effort to expose some of the mathematical structure because you will see these ideas in many places. That structure is the *inner product* on the space of *square integrable* functions.

There's much more to the structure of the Fourier coefficients and to the idea of writing a periodic function as a sum of complex exponentials than might appear from our simple derivation. There are:

- Algebraic and geometric aspects
  - The algebraic and geometric aspects are straightforward extensions of the algebra and geometry of vectors in Euclidean space. The key ideas are the inner product (dot product), orthogonality, and norm. Your job here is to transfer your intuition from geometric vectors to a more general setting where the vectors are signals; at least accept that the words transfer in some kind of meaningful way even if the pictures do not.
- Analytic aspects
  - The analytic aspects are not straightforward and require new ideas on limits and on the nature of integration. The aspect of 'analysis', as a field of mathematics distinct from other fields, is its systematic use of limiting processes. To define a new kind of limit, or to find new consequences of taking limits, is to define a new area of analysis. We really can't cover that and it's not appropriate to try to, but I'll say a little about it.

Let me introduce the notation, basic terminology and state what the important results are now, so you can see the point. Then I'll explain where these ideas come from and how they fit together.

Once again, to be definite we're working with periodic functions of period 1. We can consider such a function already to be defined for all real numbers, and satisfying the identity $f(t+1) = f(t)$ for all t, or we can consider $f(t)$ to be defined initially only on the interval from 0 to 1, say, and then extended to be periodic and defined on all of $\mathbf{R}$ by repeating the graph. In either case, once we know what we need to know about the function on $[0,1]$ we know everything. All of the action in the following discussion takes place on the interval $[0,1]$.

When $f(t)$ is a signal for $0 \leq t \leq 1$ the *energy* of the signal is defined to be the integral

$$\int_0^1 f(t)^2 \, dt.$$

This definition of energy comes up in other physical contexts also; we don't have to be talking about functions of time. Thus

$$\int_0^1 f(t)^2 \, dt < \infty$$

means that the signal has *finite energy*, a reasonable condition to expect or to impose.

For mathematical reasons, primarily, it's best to take the square root of the integral, and to define

$$||f|| = \left\{ \int_0^1 f(t)^2 \, dt \right\}^{1/2}$$

With this definition one has, for example, that

$$||\alpha f|| = |\alpha| ||f||,$$

whereas if we didn't take the square root the constant would come out to the second power – not as nice. One can also show, though the proof is not so obvious (see Appendix 1), that the triangle inequality holds:

$$||f + g|| \leq ||f|| + ||g||.$$

We also measure the distance between two functions, via

$$||f - g|| = \left\{ \int_0^1 (f(t) - g(t))^2 \, dt \right\}^{1/2}$$

Then $||f - g|| = 0$ if and only if $f = g$.

> **Now get this:** The length of a vector is the square root of the sum of the squares of its components. This integral norm is the continuous analog of that, and we'll make the analogy even closer when we introduce the corresponding dot product.

We let $L^2([0, 1])$ be the set of functions $f(t)$ on $[0, 1]$ for which

$$\int_0^1 f(t)^2 \, dt < \infty$$

The 'L' stands for Lebesgue, the French mathematician who introduced a new definition of the integral that underlies the analytic aspects of the results we're about to talk about. His work was around the turn of the 20^{th} century. The length we've just introduced, $||f||$, is called the *square norm* or the L^2*-norm* of the function. If one wants to distinguish this from other norms that might come up one writes $||f||_2$.

It's true, you'll be relieved to hear, that if $f(t)$ is in $L^2([0, 1])$ then the integral defining its Fourier coefficients exists. See Appendix 1 for this. [2]

Here now is the life's work of several generations of mathematicians, all dead, all still revered:

> Let $f(t)$ be in $L^2([0, 1])$ and let
>
> $$\hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t) \, dt, \quad n = 0, \pm 1, \pm 2, \ldots,$$
>
> be its Fourier coefficients. Then

[2] The complex integral can be written in terms of two real integrals by writing $e^{-2\pi i n t} = \cos 2\pi n t - i \sin 2\pi n t$ so everything can be defined and computed in terms of real quantities. There's something more to be said on complex-valued versus real-valued functions in all of this, but it's best to put that off just now.

(1) For any N the finite sum

$$\sum_{n=-N}^{N} \hat{f}(n)e^{2\pi nit}$$

is the best approximation in $L^2([0,1])$ (best least squares approximation, best root mean square approximation) to $f(t)$ of 'degree' N.

(2) The complex exponentials $e^{2\pi nit}$, $n = 0, \pm 1, \pm 2, \ldots$ form a basis for $L^2([0,1])$, and the partial sums, above, converge to $f(t)$ as $N \to \infty$ in the L^2-distance. This means that

$$\lim_{N\to\infty} \left\| \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi nit} - f(t) \right\| = 0$$

We write

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi nit},$$

where the equals sign is interpreted in terms of the limit.

(3) The energy can be calculated from the Fourier coefficients:

$$\int_0^1 f(t)^2 \, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

This is known, depending on who you're talking to, as Rayleigh's identity or as Parseval's theorem.

For completeness, let me add a fourth point that's a sort of converse to items two and three. We won't use this, but it ties things up nicely.

4. If $\{c_n\}$, $n = 0, \pm 1, \pm 2, \ldots$ is any sequence of complex numbers for which

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty,$$

then the function

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi nit}$$

is in $L^2([0,1])$ (meaning the limit of the partial sums converges to a function in $L^2([0,1])$) and $c_n = \hat{f}(n)$.

This last result is usually referred to as the Riesz-Fischer theorem.

And the point of this is, again ... One way to think of the formula for the Fourier coefficients is as passing from the 'time domain' to the 'frequency domain': From a knowledge of $f(t)$ (the time domain) we produce a portrait of the signal in the frequency domain, namely the (complex) coefficients $\hat{f}(n)$ associated with the (complex) harmonics $e^{2\pi nit}$. The function $\hat{f}(n)$ is defined on the integers, $n = 0, \pm 1, \pm 2, \ldots$ and the equation

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi nit},$$

recovers the time domain representation from the frequency domain representation. At least it does in the L^2 sense of equality. The extent to which equality holds in the usual, pointwise sense (plug in a value of t and the two sides agree) can be an extremely complicated and delicate question. We will avoid it – meet me on a street corner sometime and I'll tell you about it.

Recall that the set of all frequencies that occur in the Fourier series for $f(t)$ is called the *spectrum* of f. The spread of frequencies, that is the Maximum frequency−Minimum frequwncy is called the *bandwidth*; this was mentioned in the last set of notes. We'll use these terms again when we introduce the Fourier transform.

Orthogonality. The aspect of Euclidean geometry that sets it apart from other geometries which share most of its other features is perpendicularity. To set up a notion of perpendicularity is to try to copy the Euclidean properties that go with it, together with the associated reasoning. Perpendicularity becomes operationally useful when it's linked to measurement, *i.e.* length. This link is the Pythagorean theorem.[3] Perpendicularity becomes more austere when mathematicians start referring to it as *orthogonality*, but that's what I'm used to.

Vectors. To fix ideas, I want to remind you briefly of vectors and geometry in Euclidean space. We write vectors in $\mathbf{R}^n$ as n-tuples of real numbers:

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$

The v_i are called the components of $\mathbf{v}$. The length, or *norm* of $\mathbf{v}$ is

$$||\mathbf{v}|| = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}.$$

The distance between two vectors $\mathbf{v}$ and $\mathbf{w}$ is $||\mathbf{v} - \mathbf{w}||$.

How does the Pythagorean theorem look in terms of vectors? Let's just work in $\mathbf{R}^2$. Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a right triangle with $\mathbf{w}$ the hypotenuse then

$$\begin{aligned} ||\mathbf{w}||^2 = ||\mathbf{u} + \mathbf{w}||^2 &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \\ (u_1 + v_1)^2 + (u_2 + v_2)^2 &= (u_1^2 + u_2^2) + (v_1^2 + v_2^2) \\ (u_1^2 + 2u_1v_1 + v_1^2) + (u_2^2 + 2u_2v_2 + v_2^2) &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \end{aligned}$$

The squared terms cancel and we conclude that

$$u_1v_2 + u_2v_2 = 0$$

is a necessary and sufficient condition for $\mathbf{u}$ and $\mathbf{v}$ to be perpendicular.

[3]How do you lay out a big rectangular field of specified dimensions? You use the Pythagorean theorem. I had an experience with this a few summers ago when I volunteered to help lay out soccer fields. I was only asked to assist, because evidently I could not be trusted with the details. Put two stakes in to determine one side of the field. That's one leg of what is to become a right triangle – half the field. I hooked a tape measure on one stake and walked off in a direction generally perpendicular to the first leg, stopping when I had gone the regulation distance for that side of the field, or when I needed rest. The chief hooked a tape measure on the other stake and walked approximately along the diagonal of the field – the hypotenuse. We met up and adjusted our positions (I was not to change the distance I had walked) so that the Pythagorean theorem was satisfied. He had a chart showing what his distance should be. Hence the leg I determined must be perpendicular to the first leg we laid out. This was my first practical use of the Pythagorean theorem, and so began my transition from a pure mathematician to an engineer.

And so we introduce the (algebraic) definition of the *inner product*, or *dot product* of two vectors. We give this in $\mathbf{R}^n$:

> If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ then the the inner product is:
>
> $$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$
>
> Other notations for the inner product are $(\mathbf{v}, \mathbf{w})$ (just parentheses; we'll be using this notation) and $\langle \mathbf{v}, \mathbf{w} \rangle$ (angle brackets; for those who think parentheses are not fancy enough. The use of angle brackets is especially common in physics.)

Notice that

$$(\mathbf{v}, \mathbf{v}) = v_1^2 + v_2^2 + \cdots + v_n^2 = ||\mathbf{v}||^2.$$

Thus

$$||\mathbf{v}|| = (\mathbf{v}, \mathbf{v})^{1/2}.$$

There is also a geometric approach to the inner product, which leads to the alternate definition

$$(\mathbf{v}, \mathbf{w}) = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

We see that $(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. This is a truly helpful result, especially because it's so easy to verify when the vectors are given in coordinates. The inner product does more than identify orthogonal vectors, however. It also tells you how much of one vector is in the direction of another. That is, for example, the vector

$$(\mathbf{v}, \mathbf{w}) \frac{\mathbf{w}}{||\mathbf{w}||}$$

is the projection of $\mathbf{v}$ onto the unit vector $\mathbf{w}/||\mathbf{w}||$, or $(\mathbf{v}, \mathbf{w})$ is the (scalar) component of $\mathbf{v}$ in the direction of $\mathbf{w}$. In that sense I think of the inner product as measuring how much one vector 'knows' another; two orthogonal vectors don't know each other.

Finally, I want to list the main algebraic properties of the inner product, I won't give the proofs – they are straightforward verifications. We'll see these properties again, modified slightly to allow for complex numbers, a little later.

(1) $(\mathbf{v}, \mathbf{v}) \geq 0$ and $(\mathbf{v}, \mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$. (positive definiteness)
(2) $(\mathbf{v}, \mathbf{w}) = (\mathbf{w}, \mathbf{v})$ (symmetry)
(3) $(\alpha\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{v}, \mathbf{w})$ for any scalar α. (homogeneity)
(4) $(\mathbf{v} + \mathbf{w}, \mathbf{u}) = (\mathbf{v}, \mathbf{u}) + (\mathbf{w}, \mathbf{u})$ (additivity)

In fact, these are exactly the properties that ordinary multiplication has.

Orthonormal bases. The natural basis for $\mathbf{R}^n$ are the vectors of length 1 in the n 'coordinate directions':

$$\mathbf{e}_1 = (1, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots,\ \mathbf{e}_n = (0, 0, \ldots, 1).$$

It's called the 'natural' basis because, among other things, a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is expressed naturally in terms of its components as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 = \cdots + v_n\mathbf{e}_n.$$

One says that the natural basis $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$ are an *orthonormal basis* for $\mathbf{R}^n$, meaning

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij},$$

where δ_{ij} is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Notice that

$$(\mathbf{v}, \mathbf{e}_k) = v_k,$$

and hence that

$$\mathbf{v} = \sum_{k=1}^{n} (\mathbf{v}, \mathbf{e}_k)\mathbf{e}_k.$$

In words:

> $\mathbf{v}$ is decomposed as a sum of vectors in the directions of the orthonormal basis vectors, and the components are given by the inner product of $\mathbf{v}$ with the basis vectors.

Functions. All of what we've just done can be carried over to $L^2([0,1])$, even with the same motivation. When will two functions be perpendicular? If the Pythagorean theorem is satisfied. Thus if we are to have

$$||f + g||^2 = ||f||^2 + ||g||^2$$

then

$$\begin{aligned} \int_0^1 (f(t) + g(t))^2 &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt \\ \int_0^1 (f(t)^2 + 2f(t)g(t) + g(t)^2)\,dt &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt \\ \int_0^1 f(t)^2\,dt + 2\int_0^1 f(t)g(t)\,dt + \int_0^1 g(t)^2\,dt &= \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt \end{aligned}$$

If you buy the premise, you have to buy the conclusion – we conclude that the condition to adopt to define when two functions are orthogonal is

$$\int_0^1 f(t)g(t)\,dt = 0$$

So we define the inner product of two functions in $L^2([0,1])$ to be.

$$(f, g) = \int_0^1 f(t)g(t)\,dt$$

This inner product has all of the algebraic properties of the dot product of vectors. We list them, again:

(1) $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$.
(2) $(f, g) = (g, f)$
(3) $(f + g, h) = (f, h) + (g, h)$
(4) $(\alpha f, g) = \alpha(f, g)$

Now, let me relieve you of a burden that you may feel you must carry. There is no reason on earth why you should have any pictorial intuition for the inner product of two functions, and for when two functions are orthogonal. We're working by analogy here, It's a very strong analogy, but that's not to say that the situations are identical. They aren't. As I have said before, what you should do is draw pictures in $\mathbf{R}^2$ and $\mathbf{R}^3$ and *using the same words* make the attempt to carry the reasoning based on those pictures over to $L^2([0,1])$, for example.

There's a catch. There's always a catch. In the preceding discussion we've been working with the *real* vector space $\mathbf{R}^n$ and (without comment) with real-valued functions in $L^2([0,1])$. But, of course, the definition of the Fourier coefficients involves complex functions in the form of the complex exponential, and the Fourier series is a sum of complex terms. We could avoid that by writing everything in terms of sine and cosine, but not wanting to sacrifice the algebraic dexterity we can show by working with the complex form, a more effective choice is to consider *complex-valued* square integrable functions and the *complex inner product*.

Here are the definitions. For the definition of $L^2([0,1])$ we assume that

$$\int_0^1 |f(t)|^2\,dt < \infty.$$

Note the use of the magnitude $|f(t)|^2$ in the integrand, something we didn't have to do in case $f(t)$ is real-valued. The inner product of complex valued functions $f(t)$ and $g(t)$ in $L^2([0,1])$ is defined to be

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt.$$

The complex conjugate in the second slot causes a few changes in the algebraic properties. To wit:

(1) $(f,g) = \overline{(g,f)}$ (*Hermitian* symmetry)
(2) $(f,f) \geq 0$ and $(f,f) = 0$ if and only if $f = 0$ (positive definiteness – same as before)
(3) $(\alpha f, g) = \alpha(f,g)$, $(f, \alpha g) = \overline{\alpha}(f,g)$ (homogeneity – same as before in first slot, conjugate scalar comes out in second slot)
(4) $(f+g,h) = (f,h) + (g,h)$, $(f, g+h) = (f,g) + (f,h)$ (additivity – same as before, no difference between additivity in first or second slot)

From now on, when we talk about $L^2([0,1])$ and the inner product on $L^2([0,1])$ we will always assume the complex inner product. If the functions happen to be real-valued then this reduces to the earlier definition.

The complex exponentials are an orthonormal basis. One of the greatest hits of the theory of Fourier series says that the complex exponentials form a basis for $L^2([0,1])$. Particularly exciting is that, just like the natural basis of $\mathbf{R}^n$, they are an *orthonormal basis*.

Here's the calculation that demonstrates this last fact. Write

$$e_n(t) = e^{2\pi i n t}.$$

The inner product of two of them, $e_n(t)$ and $e_m(t)$, when $n \neq m$ is:

$$\begin{aligned}
(e_n, e_m) &= \int_0^1 e^{2\pi nit}\overline{e^{2\pi imt}}\,dt \\
&= \int_0^1 e^{2\pi nit}e^{-2\pi imt}\,dt \\
&= \int_0^1 e^{2\pi i(n-m)t}\,dt \\
&= \frac{1}{2\pi i(n-m)}e^{2\pi i(n-m)t}\Big]_0^1 \\
&= \frac{1}{2\pi i(n-m)}(e^{2\pi i(n-m)} - e^0) \\
&= \frac{1}{2\pi i(n-m)}(1-1) \\
&= 0
\end{aligned}$$

They are orthogonal. And when $n = m$

$$\begin{aligned}
(e_n, e_n) &= \int_0^1 e^{2\pi nit}\overline{e^{2\pi int}}\,dt \\
&= \int_0^1 e^{2\pi nit}e^{-2\pi int}\,dt \\
&= \int_0^1 e^{2\pi i(n-n)t}\,dt \\
&= \int_0^1 1\,dt = 1.
\end{aligned}$$

Therefore the functions $e_n(t)$ are *orthonormal*:

$$(e_n, e_m) = \delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

In fact, we did this very calculation when we found what the Fourier coefficients must be given that a signal has a Fourier series. Could you have made the conceptual leap from that simple fact to the realization that it's 'orthogonality' that's the unifying idea?

What is the component of a function $f(t)$ 'in the direction' $e_n(t)$? By analogy to the Euclidean case, it is given by the inner product

$$(f, e_n) = \int_0^1 f(t)e^{-2\pi int}\,dt,$$

precisely the n'th Fourier coefficient $\hat{f}(n)$.

Thus writing the Fourier series

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi nit},$$

as we did earlier, is exactly like the decomposition in terms of an orthonormal basis and associated inner product:

$$f = \sum_n (f, e_n) e_n.$$

A direct and important consequence of this is:

The Energy Theorem (also known as Rayleigh's Theorem, also known as Parseval's Identity)

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int}.$$

Then

$$\int_0^1 |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Now get this, again. Another way of writing this is

$$||f|| = \left\{ \sum_{n=-\infty}^{\infty} (f, e_n)^2 \right\}^{1/2}.$$

That is, the norm (length) of the function f is the square root of the sum of the squares of its components (f, e_n) in the orthonormal basis $e_n(t) = e^{2\pi nit}$. Maybe we should call this the Pythagorean theorem.

The proof is a cinch! With

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi nit} = \sum_{n=-\infty}^{\infty} (f, e_n) e_n.$$

we compute

$$\begin{aligned}
\int_0^1 |f(t)|^2\,dt &= ||f||^2 = (f,f) \\
&= \left(\sum_{n=-\infty}^{\infty} (f,e_n)e_n, \sum_{m=-\infty}^{\infty} (f,e_m)e_m \right) \\
&= \sum_{n,m} (f,e_n)\overline{(f,e_m)}(e_n,e_m) \\
&= \sum_{n,m=-\infty}^{\infty} (f,e_n)\overline{(f,e_m)}\delta_{nm} \\
&= \sum_{n=-\infty}^{\infty} (f,e_n)\overline{(f,e_n)} \\
&= \sum_{n=-\infty}^{\infty} |(f,e_n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2
\end{aligned}$$

Two expressions for the same thing... The Energy Theorem says that the energy

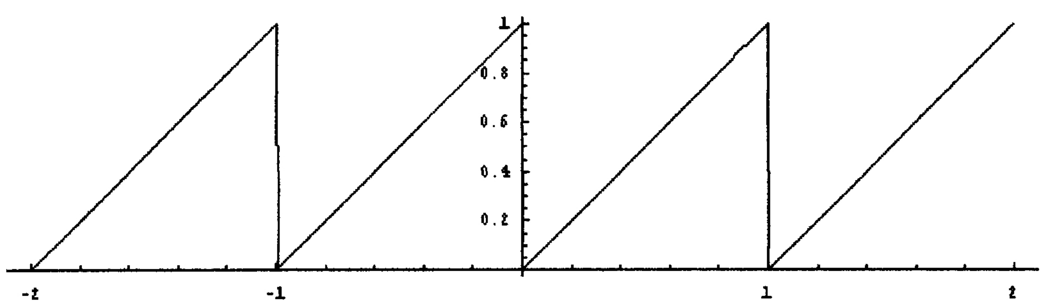
$$\int_0^1 |f(t)^2\,dt$$

can be computed from *spectral* information. It is the sum of the squares of the Fourier coefficients. We may also be able to compute the integral, of course. Then we are in the position of having two expressions for the same thing. Powerful, very powerful. Let's do an example.

Consider the sawtooth function;

$$f(t) = t, \text{for } 0 \le t < 1$$

and then extended periodically. The graph is:



We first compute the Fourier coefficients:

$$\begin{aligned}
c_0 &= \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} \\
c_n &= \int_0^1 e^{-2\pi int} t\,dt \quad (n \neq 0) \\
&= \left[-\frac{t}{2\pi in} e^{-2\pi int}\right]_0^1 - \int_0^1 -\frac{1}{2\pi in} e^{-2\pi int}\,dt \quad \text{(integrate by parts)} \\
&= -\frac{1}{2\pi in}
\end{aligned}$$

The Fourier series is then

$$S(t) = \frac{1}{2} - \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{2\pi in} e^{2\pi int}$$

and the sum of the absolute value of the squares of the coefficients is

$$\begin{aligned}
&\frac{1}{4} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{4\pi^2 n^2} \\
= \; &\frac{1}{4} + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{(combining the } \pm \text{ terms).}
\end{aligned}$$

Now let's calculate the integral of the square of the function:

$$\int_0^1 |S(t)|^2\,dt = \int_0^1 t^2\,dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}.$$

Invoking the Energy Theorem then gives

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

that is

$$\frac{1}{12} = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

That's a very famous identity. I hope you enjoyed it.

Appendix 1: The Cauchy-Schwarz inequality and its consequences

The Cauchy-Schwarz inequality is between the inner product of two vectors and their norms. It states

$$|(\mathbf{v}, \mathbf{w})| \leq ||\mathbf{v}||\,||\mathbf{w}||.$$

This is trivial to see from the geometric definition of the inner product, for

$$|(\mathbf{v}, \mathbf{w})| = ||\mathbf{v}||\,||\mathbf{w}||\,|\cos\theta| \leq ||\mathbf{v}||\,||\mathbf{w}||,$$

because $|\cos\theta| \leq 1$. In fact, the geometric definition of the inner product will *follow* from the Cauchy-Schwarz inequality.

It's certainly not trivial to derive the inequality from the algebraic definition. Written out in components, the inequality says that

$$\left|\sum_{k=1}^{n} v_k w_k\right| \leq \left\{\sum_{k=1}^{n} v_k^2\right\}^{1/2} \left\{\sum_{k=1}^{n} w_k^2\right\}^{1/2}.$$

Sit down and try that one out, sometime.

In fact, the proof of the Cauchy-Schwarz inequality in general uses only the four algebraic properties of the inner product listed earlier. Consequently the same argument applies to any sort of 'product' satisfying these properties. It's such an elegant argument (due to John von Neumann, I believe) that I'd like to show it to you. I'll give the argument for *real* inner products. I'll let you see how to modify it in the complex case.

Any inequality can ultimately be written in a way that says that some quantity is positive. There aren't many things that we know are positive: the square of a real number; the area of something; and the length of something are examples.[4] For this proof we use that the norm of a vector is positive, but we throw in a parameter.[5] Let t be any real number. Then $||\mathbf{v} - t\mathbf{w}||^2 \geq 0$. Write this in terms of the inner product and expand out using the properties, above. (Because of homogeneity and additivity, it's just like multiplication – that's important to realize):

$$\begin{aligned} 0 &\leq ||\mathbf{v} - t\mathbf{w}||^2 \\ &= (\mathbf{v} - t\mathbf{w}, \mathbf{v} - t\mathbf{w}) \\ &= (\mathbf{v}, \mathbf{v}) - 2t(\mathbf{v}, \mathbf{w}) + t^2(\mathbf{w}, \mathbf{w}) \\ &= ||\mathbf{v}||^2 - 2t(\mathbf{v}, \mathbf{w}) + t^2||\mathbf{w}||^2 \end{aligned}$$

This is a quadratic equation in t, of the form $at^2 + bt + c$, where $a = ||\mathbf{w}||^2$, $b = -2(\mathbf{v}, \mathbf{w})$, and $c = ||\mathbf{v}||^2$. The first inequality, and the chain of equalities that follow, says that this quadratic is *always non-negative*. Now a quadratic that's always non-negative has to have a *non-positive* discriminant: The discriminant, $b^2 - 4ac$ determines the nature of the roots of the quadratic. If the discriminant is positive then there are two real roots. But if there are two real roots, then the quadratic must be negative somewhere – draw a graph!

Therefore $b^2 - 4ac \leq 0$, which translates to

$$\begin{aligned} &4(\mathbf{v}, \mathbf{w})^2 - 4||\mathbf{w}||^2\,||\mathbf{v}||^2 \leq 0, \quad \text{or} \\ &(\mathbf{v}, \mathbf{w})^2 \leq ||\mathbf{w}||^2\,||\mathbf{v}||^2. \end{aligned}$$

Take the square root of both sides to obtain

$$|(\mathbf{v}, \mathbf{w})| \leq ||\mathbf{v}||\,||\mathbf{w}||,$$

as desired. (Amazing, isn't it – a non-trivial application of the *quadratic formula*!) This proof also shows when equality holds in the Cauchy-Schwarz inequality. When is that?

---

[4]This little riff on the nature of inequalities classifies as a minor secret of the universe. More subtle inequalities sometimes rely on convexity, as in the center of gravity of a system of masses is contained within the convex hull of the masses.

[5]'Throwing in a parameter' goes under the heading of dirty tricks of the universe.

We now know that

$$-1 \leq \frac{(\mathbf{v}, \mathbf{w})}{||\mathbf{v}||\,||\mathbf{w}||} \leq 1.$$

Therefore there is a unique angle θ with $0 \leq \theta \leq \pi$ such that

$$\cos\theta = \frac{(\mathbf{v}, \mathbf{w})}{||\mathbf{v}||\,||\mathbf{w}||},$$

i.e.

$$(\mathbf{v}, \mathbf{w}) = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta.$$

Identifying θ as the angle between $\mathbf{v}$ and $\mathbf{w}$ we have now reproduced the geometric definition of the inner product. What a relief.

The triangle inequality,

$$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$$

follows directly from the Cauchy-Schwarz inequality. Here's the argument:

$$\begin{aligned}
||\mathbf{v} + \mathbf{w}||^2 &= (\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) \\
&= (\mathbf{v}, \mathbf{v}) + 2(\mathbf{v}, \mathbf{w}) + (\mathbf{w}, \mathbf{w}) \\
&\leq (\mathbf{v}, \mathbf{v}) + 2|(\mathbf{v}, \mathbf{w})| + (\mathbf{w}, \mathbf{w}) \\
&\leq (\mathbf{v}, \mathbf{v}) + 2||\mathbf{v}||\,||\mathbf{w}|| + (\mathbf{w}, \mathbf{w}) \quad \text{(by Cauchy-Schwarz)} \\
&= ||\mathbf{v}||^2 + 2||\mathbf{v}||\,||\mathbf{w}|| + ||\mathbf{w}||^2 \\
&= (||\mathbf{v}|| + ||\mathbf{w}||)^2.
\end{aligned}$$

Now take the square root of both sides to get $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$. In coordinates this says that

$$\left\{\sum_{k=1}^{n}(v_k + w_k)^2\right\}^{1/2} \leq \left\{\sum_{k=1}^{n} v_k^2\right\}^{1/2} + \left\{\sum_{k=1}^{n} w_k^2\right\}^{1/2}.$$

For the inner product on $L^2([0,1])$ the Cauchy-Schwarz inequality takes the impressive form

$$\left|\int_0^1 f(t)g(t)\,dt\right| \leq \left\{\int_0^1 |f(t)|^2\,dt\right\}^{1/2} \left\{\int_0^1 |g(t)|^2\,dt\right\}^{1/2}.$$

You can think of this as a limiting case of the Cauchy-Schwarz inequality for vectors – sums of products become integrals of products on taking limits, an ongoing theme – but it's better to think in terms of general inner products and their properties. For example, we now also know that

$$||f + g|| \leq ||f|| + ||g||,$$

i.e. that

$$\left\{\int_0^1 (f(t) + g(t))^2\,dt\right\}^{1/2} \leq \left\{\int_0^1 f(t)^2\,dt\right\}^{1/2} + \left\{\int_0^1 g(t)^2\,dt\right\}^{1/2}.$$

Once again, one could, I suppose, derive this from the corresponding inequality for sums, but why keep gong through that extra work?

Incidentally, I have skipped over something here. If $f(t)$ and $g(t)$ are square integrable, then in order to get the Cauchy-Schwarz inequality working one has to know that the inner product (f, g) makes sense, *i.e.* that

$$\int_0^1 f(t)g(t)\,dt < \infty.$$

To deduce this you can first observe that[6]

$$f(t)g(t) \le f(t)^2 + g(t)^2.$$

With this

$$\int_0^1 f(t)g(t)\,dt \le \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt < \infty,$$

since we started by assuming that $f(t)$ and $g(t)$ are square integrable.

Another consequence of this last argument is the fortunate fact that the Fourier coefficients of a function in $L^2([0,1])$ exist. That is, we're worried about the existence of

$$\int_0^1 e^{-2\pi int} f(t)\,dt.$$

Now,

$$\left|\int_0^1 e^{-2\pi int} f(t)\,dt\right| \le \int_0^1 |e^{-2\pi int} f(t)|\,dt = \int_0^1 f(t)\,dt,$$

so we're worried whether

$$\int_0^1 |f(t)|\,dt < \infty,$$

i.e. that $f(t)$ is *absolutely integrable* given that it is *square* integrable. But $f(t) = f(t) \cdot 1$, and both $f(t)$ and the constant function 1 are square integrable on $[0,1]$. No worries.

> **Warning:** This casual argument *would not work* if the interval $[0,1]$ were replaced by the entire real line. The constant function 1 has an infinite integral on $\mathbf{R}$. You may think we can get around this little inconvenience, but it is *exactly* the sort of trouble that comes up in trying to apply Fourier *series* ideas (where functions are defined on finite intervals) to Fourier *transform* ideas (where functions are defined on all of $\mathbf{R}$). We are truly worried about this.

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[6]And where does that come from? From the same positivity trick used to prove Cauchy-Schwarz:

$$0 \le (f(t) - g(t))^2 = f(t)^2 - 2f(t)g(t) + g(t)^2,$$

hence

$$2f(t)g(t) \le f(t)^2 + g(t)^2.$$

This is the inequality between the arithmetic and geometric mean.